Classical Limits of Euclidean Gibbs States for Quantum Lattice Models

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Abstract

Models of quantum and classical particles on the d–dimensional lattice $\mathbb{Z}^d$ with pair interparticle interactions are considered. The classical model is obtained from the corresponding quantum one when the reduced physical mass of the particle $m = \mu/\hbar^2$ tends to infinity. For these models, it is proposed to define the convergence of the Euclidean Gibbs states, when $m \to +\infty$, by the weak convergence of the

corresponding local Gibbs specifications, determined by conditional Gibbs measures. In fact it is proved that all conditional Gibbs measures of the quantum model weakly converge to the conditional Gibbs measures of the classical model. A similar convergence of the periodic Gibbs measures and, as a result, of the order parameters, for such models with pair interactions possessing the translation invariance, has also been proven.

1 Introduction

We consider a system of interacting particles performing one– dimensional oscillations around their equilibrium positions which form a d– dimensional lattice $\mathbb{Z}^d \subset \mathbb{R}^d$. This system serves as a base for two models. The first one is a quantum mechanical model described by the following formal Hamiltonian

$$H = -\frac{1}{2m} \sum_{k \in \mathbb{Z}^d} \Delta_k + W,$$

where Δ_k stands for the Laplacian (in our case it is simply d^2/dx_k^2, $x_k \in \mathbb{R}$) and $m > 0$ is the reduced physical mass of the particle, which one obtains dividing the physical mass by $\hbar^2$. The first term corresponds to the kinetic energy of the particles, the second one describes their potential energy including the crystalline field as well as the energy of the interparticle interaction. We consider the model where this interaction is pairwise. As a reference model, we choose the model of noninteracting harmonic oscillators. Hence the formal Hamiltonian now is

$$H = \sum_k H_k^{(0)} + \sum_k U_k(x_k) + \frac{1}{2} \sum_{j,k} J_{jk} x_j x_k, \tag{1.1}$$

where all sums are taken over the lattice $\mathbb{Z}^d$. The first term is the Hamiltonian of the reference model, i.e. the sum of the Hamiltonians of identical harmonic oscillators. Each $H_k^{(0)}$ is defined in the complex Hilbert space $L^2(\mathbb{R}, dx_k)$ (see e.g. [11]) and reads

$$H_k^{(0)} = -\frac{1}{2m} \Delta_k + \frac{1}{2} x_k^2. \tag{1.2}$$

The second term in (1.1) contributes to the crystalline field making it to be anharmonic. The third term describes the interaction between the particles.

For the matrix J_{jk}, we assume that there exists $r > 0$ such that $J_{jk} = 0$ whenever the Euclidean distance $|j - k|$ exceeds this r. We also suppose that

$$c \stackrel{\text{def}}{=} \sup_j \sum_k |J_{jk}| < \infty. \tag{1.3}$$

For the functions $U_k : \mathbb{R} \to \mathbb{R}$, we assume that all of them are continuous and that the following estimate holds, for all $k \in \mathbb{Z}^d$ and all $x \in \mathbb{R}$,

$$U_k(x) \geq \frac{1}{2}\tilde{c}x^2 + b, \quad \tilde{c} > \max\{c - 1, 0\}. \tag{1.4}$$

Here the parameter c is defined by (1.3) and $b \in \mathbb{R}$. The second model is a classical version of the model described above. Its formal Hamiltonian is

$$H^{\text{cl}} = \sum_k \left(\frac{1}{2}x_k^2 + U_k(x_k)\right) + \frac{1}{2}\sum_{j,k} J_{jk}x_jx_k, \tag{1.5}$$

which means that in this case only the potential energy W is taken into account (see [1]). Heuristically the latter Hamiltonian may be obtained from the quantum one (1.1) by passing to the limit $m \to +\infty$.

The aim of this work is to study the possible convergence of the Gibbs states of the quantum model to the Gibbs states of the classical model. It should be remarked that the convergence of certain Green functions, describing the Gibbs states, in similar models with a special choice of the interaction potentials has already been proved in [1].

For the quantum lattice models, the Gibbs states are constructed as positive functionals on the algebras of observables (see e.g. [12], [16]), in contrast to the case of classical models where they are built by means of conditional probability distributions (see [13]), which form the so called Gibbs specifications (see [14]), as measures which solve the equilibrium (Dobrushin–Lanford–Ruelle) equations. But for the quantum models with unbounded operators, which we consider here, the algebraic approach does not allow to construct such states of infinite systems. In 1975 [1] an approach to the construction of Gibbs states of quantum lattice models has been initiated. This approach uses the integration theory in path spaces (see also [2], [8]–[10], [15], [16], and [21]). Here the state of an infinite system at temperature β^{-1} is defined by a probability measure ν_β on a certain space Ω_β. As in the case of classical systems, this measure solves the equilibrium equation and the Gibbs state as a positive functional may be reconstructed, analogously to

the Euclidean quantum field theory, by means of the moments of the measure ν_β, which here are the temperature Green functions. That is why the measure ν_β is known as *the Euclidean Gibbs state* of the quantum model. In the frames of such *an Euclidean* approach, it has become possible to develop substantially the theory of Gibbs states in quantum models with unbounded operators. The additional advantage of this Euclidean approach is that the Gibbs states in quantum and in classical models may be considered in one and the same setting. This setting allows to define more precisely what does it mean that quantum Euclidean Gibbs states converge to corresponding classical Gibbs states. For a given model at given temperature, a family of Euclidean Gibbs states is the family of solutions of the equilibrium equation defined by the local Gibbs specification. This family may consist of several elements. The same holds also for the corresponding classical model. The mentioned advantage lies in the fact that both quantum and classical Gibbs states may be defined as measures on one and the same space. The best possible way to study the convergence being discussed here is to show how each element of the family of Euclidean Gibbs states converges to the corresponding classical Gibbs state. But, even in the case where the cardinalities of the families of quantum and classical Gibbs states are equal, it would be very difficult to show the convergence of Euclidean Gibbs states except perhaps for very special cases. This would be all the more so in the case where these cardinalities are different (which implies that some bifurcations of the quantum states at certain values of the reduced mass m take place). In this paper, we propose to define the convergence of Euclidean Gibbs states of a quantum model to corresponding Gibbs states of a classical model in terms of the convergence of their local Gibbs specifications. Such a convergence, as it is shown below, holds in some sence at all values of the inverse temperature, which means in particular it holds even when the mentioned cardinalities are different. The possibility of families having different cardinalities, i.e. of phase transitions to take place, for sufficiently large values of the inverse temperature β follows from the results of our recent works (see [5], [6], [9] and references therein). For background concerning the main features of our technique we refer to [3] – [6], [8], [9], [10], [15].

2 Euclidean Gibbs States

Let $\mathcal{L}$, $\mathcal{L}_{\rm fin}$ denote the set of all, respectively of all finite, subsets of $\mathbb{Z}^d$. For certain value of the inverse temperature $\beta > 0$, we consider the space of continuous functions (*temperature loops*) taking equal values at the endpoints of the interval $[0, \beta]$

$$C_\beta \stackrel{\rm def}{=} \{\omega \in C([0,\beta] \to \mathbb{R}) \mid \omega(0) = \omega(\beta)\},$$

equipped with the norm

$$|\omega|_\beta \stackrel{\rm def}{=} \sup\{|\omega(\tau)| : \ \tau \in [0,\beta]\}, \tag{2.1}$$

and with the usual Banach space structure. For $\Lambda \in \mathcal{L}$, we put

$$\Omega_{\beta,\Lambda} \stackrel{\rm def}{=} \{\omega_\Lambda = (\omega_k)_{k\in\Lambda} \mid \omega_k \in C_\beta\}, \qquad \Omega_\beta \stackrel{\rm def}{=} \Omega_{\beta,\mathbb{Z}^d}. \tag{2.2}$$

$\Omega_{\beta,\Lambda}$ will be called *the temperature loop spaces* (TLS), their elements are *the configurations of the temperature loops* (at the inverse temperature β and for the domain Λ). The TLS $\Omega_{\beta,\Lambda}$ may be equipped with the product topology and with the σ–algebra $\mathcal{B}_{\beta,\Lambda}$ generated by the cylinder subsets of $\Omega_{\beta,\Lambda}$, see e.g. [3], [14], [18], [19].

In order to have the collection of all TLS $\{\Omega_{\beta,\Lambda}, \Lambda \in \mathcal{L}\}$ ordered by inclusion, one may introduce the following embedding mappings. For $\Lambda \subset \Lambda'$, we put $\omega_\Lambda \mapsto \omega_\Lambda \times 0_{\Lambda'\setminus\Lambda} \in \Omega_{\beta,\Lambda'}$. Here 0_Λ is the zero configuration in $\Omega_{\beta,\Lambda}$, and

$$\omega_\Lambda \times \xi_{\Lambda'\setminus\Lambda} \stackrel{\rm def}{=} \zeta_{\Lambda'}, \ \ \Lambda \subset \Lambda',$$

means the configuration in $\Omega_{\beta,\Lambda'}$ such that $\zeta_k = \omega_k$ for $k \in \Lambda$, and $\zeta_k = \xi_k$ for $k \in \Lambda' \setminus \Lambda$. Having in mind such embeddings, we shall consider every configuration ω_Λ as an element of all TLS $\Omega_{\beta,\Lambda'}$, with $\Lambda \subset \Lambda'$. Along with these embeddings, we define the projections

$$\omega_\Lambda \mapsto (\omega_\Lambda)_{\Lambda'} \stackrel{\rm def}{=} (\omega_k)_{k\in\Lambda\cap\Lambda'},$$

as a configuration in $\Omega_{\beta,\Lambda'}$ such that $\omega_k = 0$, for $k \in \Lambda' \setminus \Lambda$. Obviously, then $(\omega_\Lambda)_{\Lambda'}$ is the zero configuration if $\Lambda \cap \Lambda' = \emptyset$.

It is easily seen that under the assumptions made regarding the potentials U_j and J_{jk} the following expression

$$E_{\beta,\Lambda}(\omega_\Lambda|\zeta) \quad \stackrel{\rm def}{=} \quad \sum_{j\in\Lambda} \int_0^\beta U_j(\omega_j(\tau))d\tau + \frac{1}{2} \sum_{j,k\in\Lambda} \int_0^\beta J_{jk}\omega_j(\tau)\omega_k(\tau)d\tau$$

$$+ \sum_{j\in\Lambda, k\in\Lambda^c} \int_0^\beta J_{jk}\omega_j(\tau)\zeta_k(\tau))d\tau, \quad \Lambda \in \mathcal{L}_{\text{fin}}, \tag{2.3}$$

defines a continuous function $E_{\beta,\Lambda}(.\,|\zeta):\ \Omega_{\beta,\Lambda} \longrightarrow I\!R$.

The space of temperature loops C_β may naturally be embedded into the real Hilbert space $\mathcal{H}_\beta \stackrel{\text{def}}{=} L^2([0,\beta])$ with the scalar product

$$(\omega,\xi)_\beta = \int_0^\beta \omega(\tau)\xi(\tau)d\tau. \tag{2.4}$$

Let $\mathcal{H}_{\beta,j}$, $j \in Z\!\!Z^d$ be the j–th copy of $\mathcal{H}_\beta$. For $\Lambda \in \mathcal{L}_{\text{fin}}$, we put

$$\mathcal{H}_{\beta,\Lambda} \ \stackrel{\text{def}}{=}\ \bigoplus_{j\in\Lambda}\mathcal{H}_{\beta,j} = \{\omega_\Lambda = (\omega_j)_{j\in\Lambda} \mid \omega_j \in \mathcal{H}_{\beta,j}\}. \tag{2.5}$$

For short, we omit in the sequel a subscript like j when this does not cause any ambiguities.

The scalar product in the Hilbert space $\mathcal{H}_{\beta,\Lambda}$ is

$$(\omega_\Lambda,\xi_\Lambda)_{\beta,\Lambda} = \sum_{j\in\Lambda}(\omega_j,\xi_j)_\beta, \tag{2.6}$$

and for all $\Lambda \in \mathcal{L}_{\text{fin}}$, one has $\Omega_{\beta,\Lambda} \subset \mathcal{H}_{\beta,\Lambda}$.

Let us consider the following strictly positive trace class operator on $\mathcal{H}_\beta$

$$S_\beta(m) = (-m\Delta_\beta + 1)^{-1}, \tag{2.7}$$

where Δ_β stands for the Laplace operator in $L^2([0,\beta])$ and m is, as above, the reduced physical mass of the particle. Then one can define the Gaussian measure $\gamma_\beta^{(m)}$ on $\mathcal{H}_\beta$ which has zero mean and $S_\beta(m)$ as a covariance operator. The measure $\gamma_\beta^{(m)}$ is uniquely determined by its Fourier transform

$$\int_{\mathcal{H}_\beta} \exp(i(\varphi,\omega)_\beta)\gamma_\beta^{(m)} = \exp(-\frac{1}{2}(\varphi, S_\beta(m)\varphi)_\beta), \quad \varphi \in \mathcal{H}_\beta. \tag{2.8}$$

Actually, the set of continuous loops is a set of full measure (i.e. $\gamma_\beta^{(m)}(C_\beta) = 1$), and the measure $\gamma_\beta^{(m)}$ corresponds to the oscillator bridge process of length β [21]. Then the product measure

$$\gamma_{\beta,\Lambda}^{(m)}(d\omega_\Lambda) \stackrel{\text{def}}{=} \prod_{j\in\Lambda}\gamma_\beta^{(m)}(d\omega_j), \quad \Lambda \in \mathcal{L}_{\text{fin}}, \tag{2.9}$$

is a measure defined on $\mathcal{H}_{\beta,\Lambda}$ and supported on $\Omega_{\beta,\Lambda}$.

Now let the subset $\Lambda \in \mathcal{L}_{\rm fin}$ be fixed. For the model considered, the Gibbs measure in Λ, subject to a configuration $\zeta \in \Omega_\beta$, is

$$\nu_{\beta,\Lambda}^{(m)}(d\omega_\Lambda|\zeta) \stackrel{\rm def}{=} \frac{1}{Z_{\beta,\Lambda}(\zeta)} \exp\{-E_{\beta,\Lambda}(\omega_\Lambda|\zeta)\}\gamma_{\beta,\Lambda}^{(m)}(d\omega_\Lambda), \tag{2.10}$$

defined, as $\gamma_{\beta,\Lambda}^{(m)}$, on $\mathcal{H}_{\beta,\Lambda}$ and supported on the space $\Omega_{\beta,\Lambda}$. Here

$$Z_{\beta,\Lambda}(\zeta) \stackrel{\rm def}{=} \int_{\Omega_{\beta,\Lambda}} \exp\{-E_{\beta,\Lambda}(\omega_\Lambda|\zeta)\}\gamma_{\beta,\Lambda}^{(m)}(d\omega_\Lambda), \tag{2.11}$$

is the finite volume partition function subject to the external boundary condition ζ_{Λ^c}. The conditions (1.3) and (1.4) imposed on the potentials J_{jk} and U_k provide that the function $\exp\{-E_{\beta,\Lambda}(\omega_\Lambda|\zeta)\}$ is $\gamma_{\beta,\Lambda}^{(m)}$–integrable, thus the objects introduced in (2.10), (2.11) are well–defined.

For $B \in \mathcal{B}_\beta \stackrel{\rm def}{=} \mathcal{B}_{\beta,\mathbb{Z}^d}$, let $\mathbb{1}_B$ be the indicator function of B. Introduce the family of probability kernels $\{\pi_{\beta,\Lambda}^{(m)} \mid \Lambda \in \mathcal{L}\}$

$$\pi_{\beta,\Lambda}^{(m)}(B|\zeta) \stackrel{\rm def}{=} \int_{\Omega_{\beta,\Lambda}} \mathbb{1}_B(\omega_\Lambda \times \zeta_{\Lambda^c})\nu_{\beta,\Lambda}^{(m)}(d\omega_\Lambda|\zeta), \tag{2.12}$$

which satisfy the following consistency conditions (for more details see [14]). For every $\Lambda' \in \Lambda$,

$$\begin{aligned} \pi_{\beta,\Lambda}^{(m)}\pi_{\beta,\Lambda'}^{(m)}(B|\zeta) &\stackrel{\rm def}{=} \int_{\Omega_\beta} \pi_{\beta,\Lambda}^{(m)}(d\omega|\zeta)\pi_{\beta,\Lambda'}^{(m)}(B|\omega) \\ &= \pi_{\beta,\Lambda}^{(m)}(B|\zeta). \end{aligned} \tag{2.13}$$

Definition 2.1 *A probability measure ν_β on $(\Omega_\beta, \mathcal{B}_\beta)$ is said to be an Euclidean Gibbs state of the lattice model (1.1), (1.2) at the inverse temperature β if it satisfies the Dobrushin–Lanford–Ruelle (DLR) equilibrium equation:*

$$\nu_\beta \pi_{\beta,\Lambda}^{(m)} = \nu_\beta,$$

that is

$$\int_{\Omega_\beta} \nu_\beta(d\omega)\pi_{\beta,\Lambda}^{(m)}(B|\omega) = \nu_\beta(B), \tag{2.14}$$

for all $\Lambda \in \mathcal{L}$ and $B \in \mathcal{B}_\beta$.

The class of all Euclidean Gibbs measures, i.e., the set of solutions of (2.14) is denoted $\mathcal{G}(\beta)$.

3 Quasiclassical States and Classical Limits

In this section and in the subsequent one we present the formulation of our results, referring to Section 5 for all proofs. Given $\Lambda \in \mathcal{L}$, let us consider the subset of $\Omega_{\beta,\Lambda}$ consisting of constant trajectories, that is

$$\Omega^{\rm qc}_{\beta,\Lambda} \stackrel{\rm def}{=} \{\omega_\Lambda \in \Omega_{\beta,\Lambda} \mid (\forall k \in \Lambda)\ (\forall \tau \in [0,\beta])\ \ \omega_k(\tau) = x_k \in I\!\!R\}, \tag{3.1}$$

which is isomorphic to $I\!\!R^\Lambda$. We also set

$$\Omega_\beta \supset \Omega^{\rm qc}_{\beta,\mathbb{Z}^d} \stackrel{\rm def}{=} \Omega^{\rm qc}_\beta \cong I\!\!R^{\mathbb{Z}^d}.$$

Further, for $\Lambda \in \mathcal{L}$, let $\mathcal{B}^{\rm qc}_{\beta,\Lambda}$ be the σ–algebra generated by the cylinder subsets of $\Omega^{\rm qc}_{\beta,\Lambda}$, which is isomorphic to the corresponding σ-algebra $\mathcal{B}(I\!\!R^\Lambda)$ generated by the cylinder subsets of $I\!\!R^\Lambda$ but, on the other hand, is a subalgebra of $\mathcal{B}_{\beta,\Lambda}$. For every $B \in \mathcal{B}_{\beta,\Lambda}$, let

$$\mathrm{C}(B) \stackrel{\rm def}{=} B \cap \Omega^{\rm qc}_{\beta,\Lambda}. \tag{3.2}$$

We will also write

$$\mathcal{B}^{\rm qc}_{\beta,\Lambda} \ni C \cong A \in \mathcal{B}(I\!\!R^\Lambda), \tag{3.3}$$

for the pair of subsets $C \in \mathcal{B}^{\rm qc}_{\beta,\Lambda}$, $A \in \mathcal{B}(I\!\!R^\Lambda)$ which are connected by the isomorphism mentioned above. This means that they consist of exactly those ω_Λ and x_Λ, for which $\omega_j(\tau) = x_j$, for all $\tau \in [0,\beta]$ and $j \in \Lambda$.

Consider the following Gaussian measures

$$\chi_{\beta,\Lambda}(dx_\Lambda) \stackrel{\rm def}{=} \prod_{j\in\Lambda} \chi_\beta(dx_j), \quad x_\Lambda \in I\!\!R^\Lambda, \quad \Lambda \in \mathcal{L}_{\rm fin}, \tag{3.4}$$

$$\chi_\beta(dx_j) \stackrel{\rm def}{=} \sqrt{\frac{\beta}{2\pi}} \exp\{-\frac{\beta}{2}x_j^2\} dx_j, \quad x_j \in I\!\!R. \tag{3.5}$$

For $\Lambda \in \mathcal{L}_{\rm fin}$, let $\gamma^{\rm qc}_{\beta,\Lambda}$ be the Gaussian measure on $\Omega_{\beta,\Lambda}$ such that for every $B \in \mathcal{B}_{\beta,\Lambda}$

$$\gamma^{\rm qc}_{\beta,\Lambda}(B) = \chi_{\beta,\Lambda}(A), \tag{3.6}$$

where $A \cong \mathrm{C}(B)$, which is defined by (3.2), (3.3). This means that

$$\gamma^{\rm qc}_{\beta,\Lambda}(B) = \gamma^{\rm qc}_{\beta,\Lambda}(\mathrm{C}(B)), \tag{3.7}$$

i.e., it is supported on $\mathcal{B}^{\rm qc}_{\beta,\Lambda}$.

Making use of these measures we construct the conditional Gibbs measures following the scheme (2.9) – (2.14). As in (2.10) we set

$$\nu^{\rm qc}_{\beta,\Lambda}(d\omega_\Lambda|\zeta) \stackrel{\rm def}{=} \frac{1}{Z^{\rm qc}_{\beta,\Lambda}(\zeta)} \exp\{-E_{\beta,\Lambda}(\omega_\Lambda|\zeta)\}\gamma^{\rm qc}_{\beta,\Lambda}(d\omega_\Lambda), \tag{3.8}$$

$$Z^{\rm qc}_{\beta,\Lambda}(\zeta) \stackrel{\rm def}{=} \int_{\Omega_{\beta,\Lambda}} \exp\{-E_{\beta,\Lambda}(\omega_\Lambda|\zeta)\}\gamma^{\rm qc}_{\beta,\Lambda}(d\omega_\Lambda), \tag{3.9}$$

which is defined on the same space as $\nu^{(m)}_{\beta,\Lambda}(.|\zeta)$ and with $\zeta \in \Omega_\beta$. Further, (3.7) implies

$$\nu^{\rm qc}_{\beta,\Lambda}(B) = \nu^{\rm qc}_{\beta,\Lambda}({\rm C}(B)). \tag{3.10}$$

By means of the conditional Gibbs measures (3.8), (3.9) one can define the family of probability kernels $\{\pi^{\rm qc}_{\beta,\Lambda}(.|\zeta)\}$ as well as the corresponding Euclidean Gibbs states. The family of such Euclidean Gibbs states will be denoted $\mathcal{G}^{\rm qc}(\beta)$. The members of this family will be called *quasiclassical* Gibbs states.

Now let us construct the Gibbs measures for the classical model described by the Hamiltonian (1.5). To this end we introduce a function analogous to (2.3) which defines the interparticle interaction in the classical model

$$I_\Lambda(x_\Lambda|y) \stackrel{\rm def}{=} \sum_{j\in\Lambda} U_j(x_j) + \frac{1}{2}\sum_{j,k\in\Lambda} J_{jk}x_jx_k + \sum_{j\in\Lambda, k\in\Lambda^c} J_{jk}x_jy_k, \quad \Lambda \in \mathcal{L}_{\rm fin}, \tag{3.11}$$

where $y = (y_j)_{j\in\mathbb{Z}^d} \in \mathbb{R}^{\mathbb{Z}^d}$ determines the boundary conditions outside Λ and plays here the same role as ζ in the case of Euclidean Gibbs measures. It is not difficult to prove that $I_\Lambda(.|y)$ is a continuous function on $\mathbb{R}^\Lambda$, $\Lambda \in \mathcal{L}_{\rm fin}$. A conditional Gibbs measure for the classical model is introduced as follows

$$\mu_{\beta,\Lambda}(dx_\Lambda|\xi) = \frac{1}{Y_{\beta,\Lambda}(y)} \exp\{-\beta I_\Lambda(x_\Lambda|y)\}\chi_{\beta,\Lambda}(dx_\Lambda), \tag{3.12}$$

$$Y_{\beta,\Lambda}(y) = \int_{\mathbb{R}^\Lambda} \exp\{-\beta I_\Lambda(x_\Lambda|y)\}\chi_{\beta,\Lambda}(dx_\Lambda), \tag{3.13}$$

Like in the quantum case, the family of conditional Gibbs measures $\{\mu_{\beta,\Lambda}(.|y)|\Lambda \in \mathcal{L}_{\rm fin}\}$ may be used to define the following probability kernels

$$\rho_{\beta,\Lambda}(B|y) \stackrel{\rm def}{=} \int_{\mathbb{R}^\Lambda} \mathbb{1}_B(x_\Lambda \times y_{\Lambda^c})\mu_{\beta,\Lambda}(dx_\Lambda|y), \quad B \in \mathcal{B}(\mathbb{R}^{\mathbb{Z}^d}), \tag{3.14}$$

satisfying the consistency condition analogous to (2.13). The Gibbs states of the classical model at given inverse temperature β are understood in the sense of Definition 2.1. They are the measures μ_β on the space $\mathbb{R}^{\mathbb{Z}^d}$ which satisfy the equilibrium equation

$$\int_{\mathbb{R}^{\mathbb{Z}^d}} \mu_\beta(dx)\rho_{\beta,\Lambda}(B|x) = \mu_\beta(B), \tag{3.15}$$

for all $\Lambda \in \mathcal{L}$ and $B \in \mathcal{B}(\mathbb{R}^{\mathbb{Z}^d})$. The family of Gibbs states for the classical model is denoted $\mathcal{G}^{\mathrm{cl}}(\beta)$.

In the sequel we shall use the following equivalence relation on Ω_β. We set $\zeta \sim \tilde{\zeta}$ if for every $j \in \mathbb{Z}^d$,

$$\int_0^\beta \zeta_j(\tau)d\tau = \int_0^\beta \tilde{\zeta}_j(\tau)d\tau. \tag{3.16}$$

For $y \in \mathbb{R}^{\mathbb{Z}^d}$, let $\Upsilon_\beta(y)$ stand for the equivalence class consisting of ζ such that

$$\beta^{-1}\int_0^\beta \zeta_j(\tau)d\tau = y_j, \quad j \in \mathbb{Z}^d. \tag{3.17}$$

We write $y \in \Upsilon_\beta(y)$ assuming that the former y stands for the constant loop $\omega_j(\tau) = y_j$, $j \in \mathbb{Z}^d$ and $\tau \in [0, \beta]$.

Proposition 3.1 *For every $\nu \in \mathcal{G}^{\mathrm{qc}}(\beta)$ and all $B \in \mathcal{B}_\beta$*

$$\nu(B) = \nu(\mathrm{C}(B)), \tag{3.18}$$

i.e., every quasilocal Euclidean Gibbs state is supported on the configurations consisting of constant loops.

Our first theorem establishes the relationship between the families $\mathcal{G}^{\mathrm{qc}}$ and $\mathcal{G}^{\mathrm{cl}}$

Theorem 3.1 *For every $\nu \in \mathcal{G}^{\mathrm{qc}}$, there exists $\mu \in \mathcal{G}^{\mathrm{cl}}$, such that*

$$\mu(A) = \nu(B) = \nu(\mathrm{C}(B)), \tag{3.19}$$

for all $A \in \mathcal{B}(\mathbb{R}^{\mathbb{Z}^d})$ and $B \in \mathcal{B}_\beta$, where $\mathrm{C}(B) \cong A$ in the sense of (3.3). The mapping $\nu \mapsto \mu$ (3.19) is a bijection.

In order to study the convergence of Euclidean Gibbs states of the quantum model we shall use some notions concerning the weak convergence of measures on metric spaces (see e.g. [20]). Consider a measure space $(X, \mathcal{B}(X))$, where X is a real separable metric space and $\mathcal{B}(X)$ is the Borel σ–algebra of its subsets. Let $\mathcal{M}(X)$ be the space of all probability measures defined on $\mathcal{B}(X)$. Let $C_{\rm b}(X)$ stand for the space of all bounded real valued continuous functions on X. The topology on the space $\mathcal{M}(X)$ is defined by a system of open neighborhoods of a point $\mu \in \mathcal{M}(X)$, given as follows

$$\begin{aligned} & V_\mu(f_1, \ldots, f_n; \varepsilon_1, \ldots, \varepsilon_n) = \\ & \Big\{ \nu \in \mathcal{M}(X) | \ |\int f_i d\nu - \int f_i d\mu| < \varepsilon_i, \ i = 1, \ldots, n \Big\} \end{aligned} \tag{3.20}$$

with arbitrarily chosen $n \in I\!\!N$, $\varepsilon_1, \ldots, \varepsilon_n$ in $(0, +\infty)$, and $f_1, \ldots, f_n$ in $C_{\rm b}(X)$. Such a topology is said to be *the weak topology* on $\mathcal{M}(X)$. If a net of measures $\{\mu_\alpha\}$ converges to a measure $\mu \in \mathcal{M}(X)$ in this topology, we write $\mu_\alpha \Rightarrow \mu$. This convergence holds if and only if

$$\int f d\mu_\alpha \to \int f d\mu, \quad \forall f \in C_{\rm b}(X).$$

Now we may describe the weak convergence of the Euclidean Gibbs measures when $m \to +\infty$.

Theorem 3.2 *Let $\beta > 0$, $\Lambda \in \mathcal{L}_{\rm fin}$, and $y \in I\!\!R^{\mathbb{Z}^d}$ be chosen. Then, for every $\zeta \in \Upsilon_\beta(y)$,*

$$\nu^{(m)}_{\beta,\Lambda}(.|\zeta) \Rightarrow \nu^{\rm qc}_{\beta,\Lambda}(.|\zeta) = \nu^{\rm qc}_{\beta,\Lambda}(.|y), \quad m \to +\infty. \tag{3.21}$$

Unfortunately, this convergence does not imply the convergence of the probability kernels defined by (2.12), considered as measures. In fact, for appropriate functions f, one has from the above theorem

$$\begin{aligned} \int_{\Omega_\beta} f(\omega) \pi^{(m)}_{\beta,\Lambda}(d\omega|\zeta) &= \int_{\Omega_{\beta,\Lambda}} f(\omega_\Lambda \times \zeta_{\Lambda^c}) \nu^{(m)}_{\beta,\Lambda}(d\omega_\Lambda|\zeta) \\ &\to \int_{\Omega_{\beta,\Lambda}} f(\omega_\Lambda \times \zeta_{\Lambda^c}) \nu^{\rm qc}_{\beta,\Lambda}(d\omega_\Lambda|y), \quad \zeta \in \Upsilon_\beta(y), \end{aligned} \tag{3.22}$$

which shows that the dependence on ζ_{Λ^c}, in contrast to (3.21), remains after passing to the limit. Here we may prove only a somewhat weaker result. Let

$C_{\rm b}(\Omega_\beta)$ (resp. $C_{\rm b}(\mathbb{R}^{\mathbb{Z}^d})$) stand for the set of all bounded continuous real valued functions on Ω_β (resp. $\mathbb{R}^{\mathbb{Z}^d}$) and

$$\tilde{C}_{\rm b}(\Omega_\beta) \stackrel{\rm def}{=} \{ f \in \tilde{C}_{\rm b}(\Omega_\beta) \mid f(\omega) = f(\tilde{\omega}) \}, \tag{3.23}$$

for every pair $\omega \sim \tilde{\omega}$ with the equivalence defined by (3.16). One shows easily that for every $f \in \tilde{C}_{\rm b}(\Omega_\beta)$ there exists $g \in C_{\rm b}(\mathbb{R}^{\mathbb{Z}^d})$ such that $f(\omega) = g(x)$ for $\omega \in \Upsilon_\beta(x)$. The following theorem holds.

Theorem 3.3 *For every $f \in \tilde{C}_{\rm b}(\Omega_\beta)$ and any β, $\Lambda \in \mathcal{L}_{\rm fin}$, and $\zeta \in \Upsilon_\beta(y)$*

$$\int_{\Omega_\beta} f(\omega) \pi^{(m)}_{\beta,\Lambda}(d\omega|\zeta) \rightarrow \int_{\mathbb{R}^{\mathbb{Z}^d}} g(x) \rho_{\beta,\Lambda}(dx|y)$$

when $m \rightarrow +\infty$.

Remark 3.1 *Above we have restricted ourselves to the case of one–dimensional oscillations of the particles. This was done only in order to avoid further complications of notations and to make our considerations more transparent. A generalization to the case of particles oscillating in all directions ("vector case" where x_k takes values in some $\mathbb{R}^\nu$, $\nu > 1$) can be obtained with no additional troubles.*

4 Periodic Gibbs States and Order Parameters

It is fairly well known that, for $d \geq 2$, the models considered – both quantum and classical – may undergo a phase transition when the inverse temperature β exceeds a certain value β_*. The typical feature of this phenomenon is the nonuniqueness of the Euclidean Gibbs states. More precisely, it may be proven (see e.g. [3], [6], [19]) that, for the models considered here, the class of the called tempered Gibbs measures (which are the Euclidean Gibbs states with certain, physically motivated, restrictions on moments) consists of exactly one element if the inverse temperature β is small enough. In what follows, the model considered undergoes the phase transition if there exists β_* such that for $\beta > \beta_*$, the class of tempered Gibbs measures consists of more than one element. This splitting of the class of tempered Gibbs measures,

which occurs when the inverse temperature β passes β_*, is known as the phase transition in the model. But in most nontrivial cases there are no possibilities to describe the phase transitions on this level. A much more realistic approach is based on the use of *the order parameter*, which becomes positive for $\beta > \beta_*$. The order parameter should describe the symmetry breaking. This means that the symmetry of the formal Hamiltonian, which is inherited by the unique, for $\beta < \beta_*$, tempered Gibbs measure, is no longer proper, for $\beta > \beta_*$, in the case where there exist more than one tempered Gibbs measure.

Having the weak convergence of the Euclidean Gibbs measures of the quantum model to the Gibbs measure of the classical model we may study the possible connections between the order parameters in these models. For this purpose, the most convenient objects, of the type of those considered above, are the models possessing the translation invariance. Below we deal with translation invariant models with the pair interaction, described by Hamiltonians of the type of (1.1) and (1.5). For these models, the translation invariance may be obtained if one assumes all U_k being the same function U and realizes J_{jk} as a suitable function $J(.)$ of the Euclidean distance $|j-k|$. In order for local Gibbs measures to be invariant one may impose periodic boundary conditions instead of those established by means of configurations outside of Λ. We shall now consider this construction in more details.

Let a system of particles be described by the following formal translation invariant Hamiltonians

$$H_{\text{tran}} = \sum_k H_k^{(0)} + \sum_k U(x_k) + \frac{1}{2}\sum_{j,k} J(|j-k|)x_j x_k, \tag{4.1}$$

and

$$H_{\text{tran}}^{\text{cl}} = \sum_k \left(\frac{1}{2}x_k^2 + U(x_k)\right) + \frac{1}{2}\sum_{j,k} J(|j-k|)x_j x_k, \tag{4.2}$$

which correspond to the Hamiltonians (1.1) and (1.5) respectively. Here the Hamiltonian $H_k^{(0)}$ is given by (1.2) and all sums, as before, are taken over the whole lattice $\mathbb{Z}^d$. We also suppose that the function J vanishes when its argument exceeds some $r > 0$, and, in addition, U and J are assumed to obey the conditions (1.3), (1.4). Consider now a box $\Lambda \in \mathcal{L}_{\text{fin}}$

$$\Lambda = \{k = (k_1, \ldots, k_d) \in \mathbb{Z}^d \quad | \quad k_l^{(0)} \le k_l \le k_l^{(1)}, \quad l = 1, \ldots, d\},$$
$$k_l^{(0)} < k_l^{(1)}; \quad k_l^{(0)}, k_l^{(1)} \in \mathbb{Z}^d \ .$$

Given Λ and $l = 1, \ldots, d$, set

$$|j_l - k_l|_\Lambda \stackrel{\rm def}{=} \min\{|j_l - k_l|;\ k_l^{(1)} - k_l^{(0)} + 1 - |j_l - k_l|\ \}, \tag{4.3}$$

and

$$|j - k|_\Lambda^2 \stackrel{\rm def}{=} \sum_{l=1}^{d} |j_l - k_l|_\Lambda^2, \tag{4.4}$$

which defines the periodic metric in Λ. For this metric, one observes that

$$|j - k|_\Lambda \leq |j - k|. \tag{4.5}$$

Chosen Λ and β, we introduce the following continuous real valued function on $\Omega_{\beta,\Lambda}$

$$E_{\beta,\Lambda}^{\rm per}(\omega_\Lambda) \stackrel{\rm def}{=} \sum_{j\in\Lambda} \int_0^\beta U(\omega_j(\tau) d\tau + \frac{1}{2} \sum_{j,k\in\Lambda} \int_0^\beta J(|j-k|_\Lambda)\omega_j(\tau)\omega_k(\tau) d\tau, \tag{4.6}$$

which will be used to construct the periodic local Gibbs measures instead of the function (2.3). Thereby we define

$$\nu_{\beta,\Lambda}^{\rm per}(d\omega_\Lambda) \stackrel{\rm def}{=} \frac{1}{Z_{\beta,\Lambda}^{\rm per}} \exp\{-E_{\beta,\Lambda}^{\rm per}(\omega_\Lambda)\} \gamma_{\beta,\Lambda}^{(m)}(d\omega_\Lambda), \tag{4.7}$$

Here

$$Z_{\beta,\Lambda}^{\rm per} \stackrel{\rm def}{=} \int_{\Omega_{\beta,\Lambda}} \exp\{-E_{\beta,\Lambda}^{\rm per}(\omega_\Lambda)\} \gamma_{\beta,\Lambda}^{(m)}(d\omega_\Lambda), \tag{4.8}$$

and the Gaussian measure $\gamma_{\beta,\Lambda}^{(m)}$ is the same as in (2.10). Furthermore, the quasiclassical periodic Gibbs measure is defined by (4.7) with $\gamma_{\beta,\Lambda}^{\rm qc}$ (3.6) instead of $\gamma_{\beta,\Lambda}^{(m)}$:

$$\nu_{\beta,\Lambda}^{\rm qcp}(d\omega_\Lambda) = \frac{1}{Z_{\beta,\Lambda}^{\rm qcp}} \exp\{-E_{\beta,\Lambda}^{\rm per}(\omega_\Lambda)\} \gamma_{\beta,\Lambda}^{\rm qc}(d\omega_\Lambda), \tag{4.9}$$

$$Z_{\beta,\Lambda}^{\rm qcp} = \int_{\Omega_{\beta,\Lambda}} \exp\{-E_{\beta,\Lambda}^{\rm per}(\omega_\Lambda)\} \gamma_{\beta,\Lambda}^{qc}(d\omega_\Lambda), \tag{4.10}$$

Then a version of Theorem 3.2, for the periodic Gibbs measures, reads as follows.

Theorem 4.1 *Let $\beta > 0$, $\Lambda \in \mathcal{L}_{\rm fin}$ be chosen, Then for the periodic Gibbs measures (4.7), (4.9),*

$$\nu^{\rm per}_{\beta,\Lambda} \Rightarrow \nu^{\rm qcp}_{\beta,\Lambda}, \quad m \to +\infty.$$

The classical analog of (4.6) is

$$I^{\rm per}_{\Lambda}(x_\Lambda) \stackrel{\rm def}{=} \sum_{j\in\Lambda} U(x_j + \frac{1}{2}\sum_{j,k\in\Lambda} J(|j-k|_\Lambda) x_j x_k, \tag{4.11}$$

which we use to construct the classical periodic Gibbs measure

$$\mu^{\rm per}_{\beta,\Lambda}(dx_\Lambda) = \frac{1}{Y^{\rm per}_{\beta,\Lambda}} \exp\{-\beta I^{\rm per}_{\Lambda}(x_\Lambda)\}\chi_{\beta,\Lambda}(dx_\Lambda), \tag{4.12}$$

where

$$Y^{\rm per}_{\beta,\Lambda} = \int_{I\!\!R^\Lambda} \exp\{-\beta I^{\rm per}_{\Lambda}(x_\Lambda)\}\chi_{\beta,\Lambda}(dx_\Lambda), \tag{4.13}$$

and the Gaussian measure $\chi_{\beta,\Lambda}$ is defined by (3.4), (3.5). Now we introduce the order parameters which become positive for $\beta > \beta_*$, manifesting the appearance of the long range order. From now on, in addition to the previous assumptions, we assume that the anharmonic potential U is an even function, which means that the symmetry being broken is Z_2. Then the corresponding order parameters are defined by means of the following moments of the periodic Gibbs measures

$$P_\Lambda(m) \stackrel{\rm def}{=} \int_{\Omega_{\beta,\Lambda}} \left(\frac{1}{|\Lambda|}\int_0^\beta \sum_{j\in\Lambda} \omega_j(\tau)d\tau \right)^2 \nu^{\rm per}_{\beta,\Lambda}(d\omega), \tag{4.14}$$

$$Q_\Lambda \stackrel{\rm def}{=} \int_{I\!\!R^\Lambda} \left(\frac{1}{|\Lambda|}\sum_{j\in\Lambda} x_j \right)^2 \mu^{\rm per}_{\beta,\Lambda}(dx_\Lambda). \tag{4.15}$$

Here $|\Lambda|$ stands for the cardinality of Λ. In this case Theorem 4.1 implies.

Corollary 4.1 *Let $\beta > 0$, $\Lambda \in \mathcal{L}_{\rm fin}$ be chosen, then*

$$\lim_{m\to+\infty} P_\Lambda(m) = Q_\Lambda. \tag{4.16}$$

Another relation between the moments $P_\Lambda(m)$ and Q_Λ may be established for a special choice of the function U and under additional conditions imposed on the function J.

Proposition 4.1 *For the considered quantum model, let J be a nonnegative monotone decreasing function and U have the following form*

$$U(x_j) = ax_j^2 + \sum_{l=2}^{p} b_l x_j^{2l}, \quad a \in \mathbb{R},\ b_p > 0,\ b_l \geq 0,\ l = 2, \ldots, p,\ p \geq 2. \quad (4.17)$$

Then for every $\Lambda \in \mathcal{L}_{\rm fin}$, the moment $P_\Lambda(m)$ is a monotone increasing function of $m \in (0, +\infty)$, i.e., for arbitrary $m' > m$,

$$P_\Lambda(m') \geq P_\Lambda(m).$$

The proof of this assertion is based on the properties of J, U, it will be done in a separate work [7].

Now we define (see e.g. [17])

$$P(m) \stackrel{\rm def}{=} \lim_{\Lambda \nearrow \mathbb{Z}^d} P_\Lambda(m) \quad (4.18)$$

$$Q \stackrel{\rm def}{=} \lim_{\Lambda \nearrow \mathbb{Z}^d} Q_\Lambda, \quad (4.19)$$

which are the order parameters for the translation invariant quantum and classical models respectively. In fact, in order to prove the appearance of the long range order one does not need to find these limits explicitly. It is enough to show that the sequences $\{P_\Lambda(m)\}$, $\{Q_\Lambda\}$ are uniformly, with respect to Λ, below bounded. Combining these relations one concludes that in this case, the uniform boundedness, when $\Lambda \nearrow \mathbb{Z}^d$, of the sequence

$$P_\Lambda(m) \geq p(m) > 0,$$

implies the appearance of long range order for all $m' > m$, as well as for $m = +\infty$, that means in the classical model.

5 The Proofs

The proof of all our theorems is based on the following lemma, which is proven in the final part of this section.

Lemma 5.1 *For every $\Lambda \in \mathcal{L}_{\rm fin}$, $\beta > 0$, $\gamma_{\beta,\Lambda}^{(m)} \Rightarrow \gamma_{\beta,\Lambda}^{\rm cl}$.*

Proof of Proposition 3.1. We set

$$\mathcal{C} \stackrel{\mathrm{def}}{=} \{B = B_\Lambda \times \Omega_{\beta,\Lambda^c} \mid B_\Lambda \in \mathcal{B}_{\beta,\Lambda},\ \Lambda \in \mathcal{L}_{\mathrm{fin}}\}. \tag{5.1}$$

$$\mathcal{C}_{\mathrm{cl}} \stackrel{\mathrm{def}}{=} \{A = A_\Lambda \times I\!\!R^{\Lambda^c} \mid A_\Lambda \in \mathcal{B}(I\!\!R^\Lambda),\ \Lambda \in \mathcal{L}_{\mathrm{fin}}\}. \tag{5.2}$$

By the definition of the probability kernels (2.12), which is also valid for the quasiclassical ones,

$$\pi^{\mathrm{qc}}_{\beta,\Lambda}(B_\Lambda \times \Omega_{\beta,\Lambda^c}|\omega) = \nu^{\mathrm{qc}}_{\beta,\Lambda}(B_\Lambda|\omega_{\Lambda^c}). \tag{5.3}$$

Since ν is in $\mathcal{G}^{\mathrm{qc}}(\beta)$ it obeys the equilibrium equation (2.14) with the quasiclassical kernels $\pi^{\mathrm{qc}}_{\beta,\Lambda}$. Let some $B \in \mathcal{C}$ be chosen. Then it is a cylinder $B_\Lambda \times \Omega_{\beta,\Lambda^c}$ with certain $\Lambda \in \mathcal{L}_{\mathrm{fin}}$, thus one can choose in (2.14) this Λ. This and (5.3) yield

$$\begin{aligned} \nu(B_\Lambda \times \Omega_{\beta,\Lambda^c}) &= \int_{\Omega_\beta} \pi^{\mathrm{qc}}_{\beta,\Lambda}(B_\Lambda \times \Omega_{\beta,\Lambda^c}|\omega)\nu(d\omega) \qquad (5.4)\\ = &\int_{\Omega_\beta} \nu^{\mathrm{qc}}_{\beta,\Lambda}(B_\Lambda|\omega_{\Lambda^c})\nu(d\omega) = \int_{\Omega_\beta} \nu^{\mathrm{qc}}_{\beta,\Lambda}(\mathrm{C}(B_\Lambda)|\omega_{\Lambda^c})\nu(d\omega), \end{aligned}$$

which follows from (3.10). Thus

$$\nu(B_\Lambda \times \Omega_{\beta,\Lambda^c}) = \nu(\mathrm{C}(B_\Lambda) \times \Omega_{\beta,\Lambda^c}), \quad \Lambda \in \mathcal{L}_{\mathrm{fin}}.$$

This implies (3.18) ■

Proof of Theorem 3.1. Comparing (2.3) and (3.11) one concludes that for every $\zeta \in \Upsilon_\beta(y)$ and $\omega_\Lambda \in \Omega^{\mathrm{qc}}_{\beta,\Lambda}$, such that $\omega_k(\tau) = x_k$, $k \in \Lambda$

$$E_{\beta,\Lambda}(\omega_\Lambda|\zeta) = \beta I_\Lambda(x_\Lambda|y). \tag{5.5}$$

This and (3.10) imply for such ζ

$$\nu^{\mathrm{qc}}_{\beta,\Lambda}(B_\Lambda|\zeta) = \nu^{\mathrm{qc}}_{\beta,\Lambda}(\mathrm{C}(B_\Lambda)|\zeta) = \mu_{\beta,\Lambda}(A_\Lambda|y), \tag{5.6}$$

where $\mathrm{C}(B_\Lambda) \cong A$. Now let us define on $\mathcal{C}_{\mathrm{cl}}$ the following cylinder measure

$$\mu(A) = \mu(A_\Lambda \times I\!\!R^{\Lambda^c}) \stackrel{\mathrm{def}}{=} \nu(\mathrm{C}(B_\Lambda) \times \Omega_{\beta,\Lambda^c}), \quad \mathrm{C}(B_\Lambda) \cong A. \tag{5.7}$$

Then

$$\begin{aligned} \mu(A) &= \mu(A_\Lambda \times I\!\!R^{\Lambda^c}) = \int_{\Omega_\beta} \nu^{\mathrm{qc}}_{\beta,\Lambda}(\mathrm{C}(B_\Lambda)|\omega_{\Lambda^c})\nu(d\omega) \qquad (5.8)\\ &= \int_{\Omega^{\mathrm{qc}}_\beta} \nu^{\mathrm{qc}}_{\beta,\Lambda}(\mathrm{C}(B_\Lambda)|\omega_{\Lambda^c})\nu(d\omega) = \int_{I\!\!R^{\mathbb{Z}^d}} \mu_{\beta,\Lambda}(A_\Lambda|x_{\Lambda^c})\nu(dx). \end{aligned}$$

Here we have taken into account that the measure ν has $\Omega_\beta^{\mathrm{qc}}$ as support. Since μ is defined by a measure, it can be continued as a measure on the whole σ-algebra $\mathcal{B}(I\!\!R^{\mathbb{Z}^d})$. Directly from (5.8) one sees that this measure obeys the equilibrium equation (3.15) thus it belongs to $\mathcal{G}^{\mathrm{cl}}$. Now for every $\mu \in \mathcal{G}^{\mathrm{cl}}$, one can define a cylinder measure on $\mathcal{C}$ as given by (5.1) by a relation of the type of (5.7) and repeat the above steps obtaining an element of $\mathcal{G}^{\mathrm{qc}}$. ■

Proof of Theorem 3.2. We remind that in the case considered the function $E_{\beta,\Lambda}(\omega_\Lambda|\zeta)$ is given by (2.3). Then the density

$$F_{\beta,\Lambda}(\omega_\Lambda|\zeta) \stackrel{\mathrm{def}}{=} \frac{\nu^{(m)}_{\beta,\Lambda}(d\omega_\Lambda|\zeta)}{\gamma^{(m)}_{\beta,\Lambda}(d\omega_\Lambda)}$$

may be written as

$$F_{\beta,\Lambda}(\omega_\Lambda|\zeta) = \frac{1}{Z_{\beta,\Lambda}(\zeta)} \exp\left\{ - \sum_{j\in\Lambda, k\in\Lambda^c} J_{jk} \int_0^\beta \omega_j(\tau)\zeta_k(\tau)d\tau \right\} \Psi_{\beta,\Lambda}(\omega_\Lambda),$$

with a certain $\Psi_{\beta,\Lambda} \in C_{\mathrm{b}}(\Omega_{\beta,\Lambda})$. Therefore, for an arbitrary function $G \in C_{\mathrm{b}}(\Omega_{\beta,\Lambda})$, one has

$$\begin{aligned}
& \int_{\Omega_{\beta,\Lambda}} G(\omega_\Lambda)\nu^{(m)}_{\beta,\Lambda}(d\omega_\Lambda|\zeta) = \frac{1}{Z_{\beta,\Lambda}(\zeta)} \int_{\Omega_{\beta,\Lambda}} G(\omega_\Lambda)\Psi_{\beta,\Lambda}(\omega_\Lambda) \\
& \exp\left\{ - \sum_{j\in\Lambda, k\in\Lambda^c} J_{jk} \int_0^\beta \omega_j(\tau)\zeta_k(\tau)d\tau \right\} \gamma^{(m)}_{\beta,\Lambda}(d\omega_\Lambda) \\
\longrightarrow \quad & \frac{1}{Z_{\beta,\Lambda}(\zeta)} \int_{\Omega_{\beta,\Lambda}} G(\omega_\Lambda)\Psi_{\beta,\Lambda}(\omega_\Lambda) \\
& \exp\left\{ - \sum_{j\in\Lambda, k\in\Lambda^c} J_{jk} \int_0^\beta \omega_j(\tau)\zeta_k(\tau)d\tau \right\} \gamma^{\mathrm{qc}}_{\beta,\Lambda}(d\omega_\Lambda) \\
= \quad & \frac{1}{Z_{\beta,\Lambda}(\zeta)} \int_{I\!\!R^\Lambda} G(x_\Lambda)\Psi_{\beta,\Lambda}(x_\Lambda) \exp\left\{ - \sum_{j\in\Lambda, k\in\Lambda^c} J_{jk}x_j \int_0^\beta \zeta_k(\tau)d\tau \right\} \chi_{\beta,\Lambda}(dx_\Lambda) \\
= \quad & \frac{1}{Z_{\beta,\Lambda}(\zeta)} \int_{\Omega_{\beta,\Lambda}} G(\omega_\Lambda)\Psi_{\beta,\Lambda}(\omega_\Lambda) \exp\left\{ - \sum_{j\in\Lambda, k\in\Lambda^c} \beta J_{jk}\omega_j(0)y_k \right\} \gamma^{\mathrm{qc}}_{\beta,\Lambda}(d\omega_\Lambda) \\
= \quad & \int_{I\!\!R^\Lambda} G^{\mathrm{cl}}(x_\Lambda)\mu_{\beta,\Lambda}(d\omega_\Lambda|\xi),
\end{aligned}$$

where G^{qc} is a restriction of G on $I\!\!R^\Lambda \cong \Omega^{\mathrm{qc}}_{\beta,\Lambda}$ and we have taken into account that $\zeta \in \Upsilon_\beta(y)$. ■

The proof of Theorem 4.1 may be performed by a repetition of the arguments just used.

Proof of Theorem 3.3. It follows directly from (3.22) and (5.6). ■

Now it remains to prove Lemma 5.1. To this end we use the following known property of Gaussian measures on a Hilbert space $\mathcal{H}$ (see e.g. pp. 153–155 of book [20]).

Proposition 5.1 *Let a net of Gaussian measures* $\{\gamma_\alpha\}$ *on a separable real Hilbert space* $\mathcal{H}$ *be given. Let also each measure* γ_α *have zero mean and a trace class operator on* $\mathcal{H}$*,* S_α*, as a covariance operator. Suppose that the net* $\{S_\alpha\}$ *converges in the trace norm to an operator* S*. Then there exists a Gaussian measure* γ *on the space* $\mathcal{H}$ *such that the operator* S *is its covariance operator, and* $\gamma_a \Rightarrow \gamma$*.*

Proof of Lemma 5.1. First of all we construct explicitly the covariance operators of the Gaussian measures $\gamma_{\beta,\Lambda}$ and $\gamma_{\beta,\Lambda}^{\mathrm{qc}}$ defined by (2.9) and (3.6) respectively. The former one implies

$$S_{\beta,\Lambda}(m) = \sum_{j\in\Lambda} S_{\beta,j}(m) P_{\beta,j}, \tag{5.9}$$

where $P_{\beta,j}$ is the projector from $\mathcal{H}_{\beta,\Lambda}$ onto the space $\mathcal{H}_{\beta,j}$. Here, as before, we omit the subscript j when this does not cause any ambiguities.

In the sequel we will need a base of the Hilbert space $\mathcal{H}_\beta = \mathcal{H}_{\beta,j}$, which we choose as the following orthonormal set the eigenfunctions of Δ_β

$$\begin{aligned} e_q(\tau) &= \sqrt{\frac{2}{\beta}}\cos q\tau, \ \text{for } q>0; \quad q \in \mathcal{Q} \stackrel{\text{def}}{=} \{\frac{2\pi}{\beta}n \mid n \in \mathbb{Z}\}, \\ e_q(\tau) &= \sqrt{\frac{2}{\beta}}\sin q\tau, \ \text{for } q<0; \quad e_0(\tau) = \sqrt{\frac{1}{\beta}}. \end{aligned} \tag{5.10}$$

The operator $S_{\beta,\Lambda}(m)$ acts on $\mathcal{H}_{\beta,\Lambda}$ as a positive compact operator, hence it has the canonical representation:

$$S_{\beta,\Lambda}(m) = \sum_{j\in\Lambda}\left(\sum_{q\in\mathcal{Q}}(mq^2+1)^{-1}W_q\right)P_{\beta,j}. \tag{5.11}$$

where W_q is the projector from $\mathcal{H}_\beta$ onto the direction e_q (see (5.10)). Having this representation we prove Lemma 5.1 by showing that the net of covariance operators $\{S_{\beta,\Lambda}(m) | m \in (0,+\infty)\}$ converges in the trace norm to the

covariance operator of the measure $\gamma^{\rm qc}_{\beta,\Lambda}$ defined by (3.6), (3.4), (3.5). Let us construct the covariance operator of the latter measure. To this end we write its Fourier transformation, which should have the form (2.8), valid for all Gaussian measures:

$$\Gamma^{\rm qc}_{\beta,\Lambda}(\varphi_\Lambda) \stackrel{\rm def}{=} \int_{\mathcal{H}_{\beta,\Lambda}} \exp\{i(\varphi_\Lambda, \omega_\Lambda)_{\beta,\Lambda}\}\gamma^{\rm qc}_{\beta,\Lambda}(d\omega_\Lambda) \tag{5.12}$$

$$= \exp\{-\frac{1}{2}(\varphi_\Lambda, S^{\rm qc}_{\beta,\Lambda}\varphi_\Lambda)_{\beta,\Lambda}\}, \tag{5.13}$$

where the scalar product $(.,.)_{\beta,\Lambda}$ is defined by (2.6). On the other hand, (3.6) implies that the measure $\gamma^{\rm qc}_{\beta,\Lambda}$ is supported on the subset $\Omega^{\rm qc}_{\beta,\Lambda} \subset \Omega_{\beta,\Lambda} \subset \mathcal{H}_{\beta,\Lambda}$, where it coincides with the measure $\chi_{\beta,\Lambda}$ given by (3.4), (3.5). This yields in (5.12)

$$\begin{aligned}\Gamma^{\rm qc}_{\beta,\Lambda}(\varphi_\Lambda) \;=\; & \left(\frac{\beta}{2\pi}\right)^{\frac{|\Lambda|}{2}} \int_{I\!\!R^\Lambda} \exp\left\{ i \sum_{j\in\Lambda} x_j \int_0^\beta \varphi_j(\tau)d\tau \right\} \\ & \exp\left\{-\frac{\beta}{2}\sum_{j\in\Lambda} x_j^2\right\} \prod_{j\in\Lambda} dx_j \\ & = \exp\left\{-\frac{1}{2\beta}\sum_{j\in\Lambda}\left(\int_0^\beta \varphi_j(\tau)d\tau\right)^2\right\} \\ & = \exp\left\{-\frac{1}{2}\sum_{j\in\Lambda}(e_0,\varphi_j)^2_\beta\right\}. \end{aligned} \tag{5.14}$$

Here we have used the eigenfunction e_0 given by (5.10). Comparing the latter form of $\Gamma^{\rm qc}_{\beta,\Lambda}$ with the definition (5.12), one concludes that

$$S^{\rm qc}_{\beta,\Lambda} = \sum_{j\in\Lambda} W_0 P_{\beta,j}, \tag{5.15}$$

where W_0 is a projector in $\mathcal{H}_\beta$ on the direction e_0. Now we use the canonical representation (5.11) and obtain

$$S_{\beta,\Lambda}(m) - S^{\rm qc}_{\beta,\Lambda} = \sum_{j\in\Lambda}\left(\sum_{q\in\mathcal{Q}\setminus\{0\}} \frac{1}{mq^2+1}W_q\right)P_{\beta,j},$$

which yields

$$\mathrm{trace}(S_{\beta,\Lambda}(m) - S^{\rm qc}_{\beta,\Lambda}) \;=\; |\Lambda| \sum_{q\in\mathcal{Q}\setminus\{0\}} \frac{1}{mq^2+1} \le |\Lambda| \sum_{q\in\mathcal{Q}\setminus\{0\}} \frac{1}{mq^2}$$

$$= \frac{1}{m}\left(\frac{|\Lambda|\beta^2}{2\pi^2}\sum_{n\in \mathbb{N}}\frac{1}{n^2}\right) \longrightarrow 0, \quad m\to +\infty.$$

■

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